Generalized tanh Method Extended to Special Types of Nonlinear Equations

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By some 'pre-possessing' techniques we extend the generalized tanh method to special types of nonlinear equations for constructing their multiple travelling wave solutions. The efficiency of the method can be demonstrated for a large variety of special equations such as those considered in this paper, double sine-Gordon equation, (2+1)-dimensional sine-Gordon equation, Dodd-Bullough-Mikhailov equation, coupled Schrödinger-KdV equation and (2+1)-dimensional coupled Davey-Stewartson equation. – Pacs: 03.40.Kf; 02.30.Jr.

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1. Introduction

The tanh method, developed for years, is one of most direct and effective algebraic method for finding exact solutions of nonlinear equations [1 - 3]. Recently, much work has been concentrated on the various extensions and applications of the method [2 - 13]. Parkes and Duffy have further developed a powerful automated tanh method, in which a Mathematica package can deal with the tedious algebraic computation and put out required solutions [2, 3]. Other generalizations have been carried out by Gao, Tian and Gudkov to find more general soliton-like solutions [4, 5]. Recently we presented a generalized tanh method for obtaining multiple travelling wave solutions [11 - 13]. The key idea is to use the solution of a Riccati equation to replace the tanh function in the tanh method. We simply describe this method as follows.

For given a nonlinear equation

$$H(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{1.1}$$

when we look for its travelling wave solutions, the first step is to introduce the wave transformation $u = U(\xi)$, $\xi = x + \lambda t$ and change (1.1) to an ordinary differential equation (ODE)

$$H(U, U', U'', \cdots) = 0. \tag{1.2}$$

The next crucial step is to introduce a new variable $\varphi = \varphi(\xi)$ which is a solution of the Riccati equation

$$\varphi' = k + \varphi^2. \tag{1.3}$$

Then we propose the following series expansion as a solution of (1.1):

$$u(x,t) = U(\xi) = \sum_{i=0}^{m} a_i \varphi^i, \tag{1.4}$$

where the positive integer m can be determined by balancing the highest derivative term with nonlinear terms in (1.2). Substituting (1.3) and (1.4) into (1.2) and then setting zero all coefficients of φ^i, we can obtain a system of algebraic equations, from which the constants $k, \lambda, a_0, \cdots, a_m$ are obtained explicitly. Fortunately, the Riccati equation admits several types of solutions:

$$\varphi = \begin{cases} -\sqrt{-k}\tanh(\sqrt{-k}\xi) \\ -\sqrt{-k}\coth(\sqrt{-k}\xi) \end{cases} \quad \text{for } k < 0, \tag{1.5}$$

$$\varphi = -\frac{1}{\xi} \qquad \text{for } k = 0, \tag{1.6}$$

$$\varphi = \begin{cases} \sqrt{k}\tan(\sqrt{k}\xi) \\ -\sqrt{k}\cot(\sqrt{k}\xi) \end{cases} \quad \text{for } k > 0. \tag{1.7}$$

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Another advantage of the Riccati equation (1.3) is that the sign of k can be used to exactly judge the amount and types of travelling wave solutions of (1.1). For example, if $k < 0$, we are sure that (1.1) admits tanh-type and coth-type solutions. Especially, (1.1) will possess five types of solutions if k is an arbitrary constant. The algorithm presented here is also a computeriable method, in which generating an algebraic system from (1.1) and solving it are two key procedures, laborious to do by hand. But they can be implemented on a computer with help of the symbolic computation software *Mathematica*. The output solutions from the algebraic system comprise a list of the form $\{\lambda, k, a_0, \cdots\}$. In general, if the values of some parameters are left unspecified, then they are regard to be arbitrary for the solution of (1.1).

In the tanh method or the above generalized tanh method, (1.1) is required to be of differential polynomial form (i. e Bugers equation, KdV equation, Boussinesq equation). But physics and engineering often provide special types of nonlinear equations, such as sine-Gordon equation, sinh-Gordon equation and Schrödinger equation, which cannot be directly solved by tanh method. The aim of this paper is to solve such equations by the above proposed generalized tanh method. As illustrative examples we shall establish a series of soliton solutions, periodic solutions and rational solutions for the double sine-Gordon equation, (2+1)-dimensional sin-Gordon equation, Dodd-Bullough-Mikhailov equation, coupled Schrödinger-KdV equation and (2+1)-dimensional Davey-Stewartson equation. For further illustrating the properties of these solutions, we choose the Dodd-Bullough-Mikhailov equation and coupled Schrödinger-KdV equation and draw their figures. The coth-type and cot-type travelling wave solutions are omitted in this paper for the simplicity, since they always appear in pairs with tanh-type and tan-type solutions.

2. Examples

Example 1: Consider the double sine-Gordon equation [14]

$$u_{xt} = \sin u + \sin 2u. \tag{2.1}$$

In order to apply the generalized tanh method described in Sect. 1, we first introduce the transformations

$$\sin u = \frac{v - v^{-1}}{2i}, \ \sin 2u = \frac{v^2 - v^{-2}}{2i}, \ v = e^{iu}$$

and change (2.1) to the following required form:

$$2vv_{xt} - 2v_x v_t - v^4 - v^3 + v + 1 = 0. \tag{2.2}$$

Again using the wave transformation $v = V(\xi)$, $\xi = x + \lambda t$, we reduce (2.2) to an ODE

$$2\lambda VV'' - 2\lambda V'^2 - V^4 - V^3 + V + 1 = 0. \tag{2.3}$$

By using the ansatz (1.3) and (1.4) and balancing the term VV'' with the term V^4 in (2.3), we obtain $m = 1$, and hence

$$v = a_0 + a_1\varphi. \tag{2.4}$$

Substituting (2.4) into (2.3) and setting all coefficients of φ^i $(i = 1, 2, \cdots)$ to zero, we further obtain the following system of algebraic equations for a_0, a_1, k and λ:

$$\begin{aligned}
&1 + a_0 - a_0^3 - a_0^4 - 2k^2\lambda a_1^2 = 0,\\
&a_1 + 4a_0 a_1 \lambda k - 3a_0^2 a_1 - 4a_0^3 a_1 = 0,\\
&-3a_0 a_1^2 - 6a_0^2 a_1^2 = 0,\\
&4a_0 a_1 \lambda - a_1^3 - 4a_0 a_1^3 = 0,\\
&2a_1^2\lambda - a_1^4 = 0.
\end{aligned}$$

From the output of *Mathematica* we get a solution, namely

$$a_0 = -\frac{1}{2}, \ \lambda = \frac{1}{2}a_1^2, \ k = \frac{3}{4a_1^2},$$

where $a_1 \neq 0$ is an arbitrary constant.

Since $k > 0$, according to (1.7) we obtain a travelling wave solution

$$u_1 = \arccos\left(\frac{(\sqrt{3}\tan\theta - 1)^2 + 4}{4(\sqrt{3}\tan\theta - 1)}\right),$$

where $\theta = \frac{\sqrt{3}}{2a_1}\xi = \frac{\sqrt{3}}{2a_1}(x + \frac{1}{2}a_1^2 t)$.

Example 2: Consider the (2+1)-dimensional sine-Gordon equation [15]

$$u_{tt} - u_{xx} - u_{yy} + m^2 \sin u = 0. \tag{2.5}$$

Similar to the above Example 1, we introduce the transformations

$$v = e^{iu}, \ v(x,t) = V(\xi), \ \xi = x + \gamma y + \lambda t$$

and change (2.5) into the required form

$$2(\lambda^2-\gamma^2-1)(VV''+V'^2)+m^2(V^3-V)=0. \quad (2.6)$$

Balancing the terms VV'' and V^3 in (2.6) may lead to the ansatz

$$V=a_0+a_1\varphi+a_2\varphi^2. \quad (2.7)$$

Substituting expansion (2.7) into (2.6) and using *Mathematica*, we get the following system of algebraic equations:

$$\begin{aligned}
&-m^2a_0+m^2a_0^3-2(\lambda^2-\gamma^2-1)k^2a_1^2\\
&\quad+4(\lambda^2-\gamma^2-1)k^2a_0a_2=0,\\
&-m^2a_1+4(\lambda^2-\gamma^2-1)k^2a_0a_1+3m^2a_0^2a_1\\
&\quad-4(\lambda^2-\gamma^2-1)k^2a_1a_2=0,\\
&3m^2a_0a_1^2-m^2a_2+16(\lambda^2-\gamma^2-1)k^2a_0a_2\\
&\quad+3m^2a_0^2a_2-4(\lambda^2-\gamma^2-1)k^2a_2^2=0,\\
&4(\lambda^2-\gamma^2-1)k^2a_0a_1-m^2a_1^3\\
&\quad+4(\lambda^2-\gamma^2-1)k^2a_1a_2+6m^2a_0a_1a_2=0,\\
&2(\lambda^2-\gamma^2-1)k^2a_1^2+12(\lambda^2-\gamma^2-1)k^2a_0a_2\\
&\quad+3m^2a_1^2a_2+3m^2a_0a_2^2=0,\\
&8(\lambda^2-\gamma^2-1)k^2a_1a_2+3m^2a_1a_2^2=0,\\
&4(\lambda^2-\gamma^2-1)k^2a_2^2+m^2a_2^3=0.
\end{aligned}$$

Solving it by *Mathematica* gives two kinds of solutions

$$a_0=a_1=0,\ \ a_2=\frac{1}{k},\ \ \gamma=\pm\frac{1}{2}\sqrt{-4+4\lambda^2+\frac{m^2}{k^2}},$$

$$a_0=a_1=0,\ \ a_2=-\frac{1}{k},\ \ \gamma=\pm\frac{1}{2}\sqrt{-4+4\lambda^2-\frac{m^2}{k^2}},$$

where $k\neq 0,\lambda$ are arbitrary constants.

By using (1.5) and (1.7) we obtain two solutions, namely,

$$u_1=\arccos\left(\frac{\tanh^4\sqrt{-k}\xi\mp 1}{2\tanh^2\sqrt{-k}\xi}\right),\text{for}\ \ k<0,$$

$$u_2=\arccos\left(\frac{\tan^4\sqrt{k}\xi\pm 1}{2\tan^2\sqrt{k}\xi}\right),\text{for}\ \ k>0,$$

where $\xi=x\pm\frac{1}{2}\sqrt{-4+4\lambda^2\pm\frac{m^2}{k^2}}y+\lambda t$.

Example 3. Consider the Dodd-Bullough-Mikhailov equation [16]

$$u_{xt}+pe^u+qe^{-2u}=0, \quad (2.8)$$

which is the Liouville equation ifn $q=0$ [17].

From the transformation $u=\ln v,\ v=V(\xi),\xi=x+\lambda t$, (2.8) becomes

$$\lambda VV''-\lambda V'^2+pV^3+q=0. \quad (2.9)$$

Balancing the terms VV'' and V^3, we obtain the following ansatz:

$$V=a_0+a_1\varphi+a_2\varphi^2. \quad (2.10)$$

Substituting (2.10) into (2.9) and using *Mathematica*, we get the following system of algebraic equations:

$$\begin{aligned}
&q+pa_0^3-\lambda k^2a_1^2+2\lambda k^2a_0a_2=0,\\
&2\lambda ka_0a_1+3pa_0^2a_1-2\lambda k^2a_1a_2=0,\\
&3pa_0a_1^2+8\lambda ka_0a_2+3pa_0^2a_2-2\lambda k^2a_2^2=0,\\
&2\lambda k^2a_0a_1+pa_1^3+2\lambda k^2a_1a_2+6pa_0a_1a_2=0,\\
&\lambda k^2a_1^2+6\lambda k^2a_0a_2+3pa_1^2a_2+3pa_0a_2^2=0,\\
&4\lambda k^2a_1a_2+3pa_1a_2^2=0,\\
&2\lambda k^2a_2^2+pa_2^3=0.
\end{aligned}$$

Solving it by *Mathematica* gives two solutions:

$$q=0,\ \ a_1=0,\ \ a_0=ka_2,\ \ \lambda=-\frac{1}{2}pa_2 \quad (2.11)$$

with a_2, k being constants, and

$$a_1=0,\ \ a_0=\frac{3q^{1/3}}{2p^{1/3}},\ \ a_2=\frac{q^{1/3}}{2p^{1/3}k},\ \ \lambda=-\frac{(p^2q)^{1/3}}{4k} \quad (2.12)$$

with $k\neq 0$ being constants.

Then from (1.5)-(1.7) and (2.11), we obtain three kinds of solutions for the Liouville equation ($q=0$):

$$\begin{aligned}
&u_1=\ln(ka_2\text{sech}^2\sqrt{-k}\xi),\ \ k<0,\\
&u_2=\ln(a_2k\sec^2\sqrt{k}\xi),\ \ k>0,\\
&u_3=\ln\frac{a_2}{\xi^2},\ \ k=0,
\end{aligned}$$

where $\xi=x-\frac{1}{2}pa_2t$.

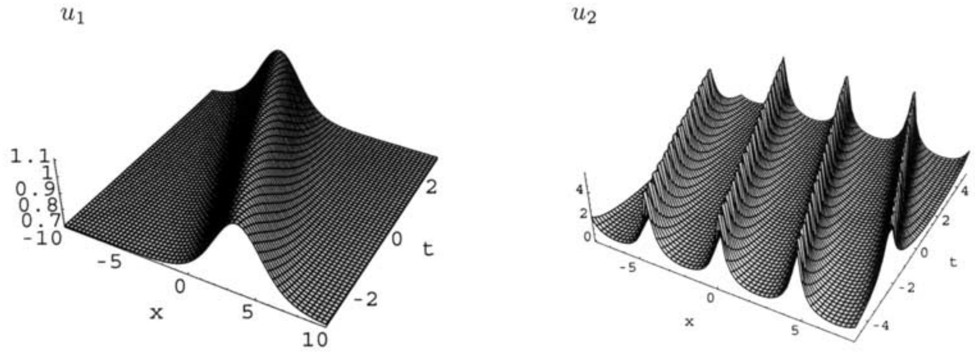

Fig. 1. The soliton solution u_1 and periodic solution u_2, where $p = q = 1, k = 0.5$.

From (1.5), (1.7) and (2.12), we get two kinds of solutions for the Dodd-Bullough-Mikhailov equation ($q \neq 0$)

$$u_1 = \ln\left[\frac{q^{1/3}}{2p^{1/3}}(3 - \tanh^2\sqrt{-k}\xi)\right], \quad k < 0,$$
$$u_2 = \ln\left[\frac{q^{1/3}}{2p^{1/3}}(3 + \tan^2\sqrt{k}\xi)\right], \quad k > 0,$$

where $\xi = x - \frac{(p^2q)^{1/3}}{4k}t$. The plots for the solutions u_1 and u_2 are given in Figure 1.

Example 4. Consider the coupled Schrödinger-KdV equation [18]

$$iu_t = u_{xx} + uv,$$
$$v_t + 6vv_x + v_{xxx} = (|u|^2)_x. \quad (2.13)$$

We introduce the transformations

$$u = e^{i\theta}U(\xi), \quad v = V(\xi), \quad \theta = \alpha x + \beta t, \quad \xi = x + \lambda t, \quad (2.14)$$

where α, β and λ are real constants and $U(\xi), V(\xi)$ are real functions.

Substituting (2.14) into (2.13) we obtain the relation $\lambda = 2\alpha$, and U, V satisfy the system

$$U'' + (\beta - \alpha^2)U + UV = 0,$$
$$2\alpha V' + 6VV' + V''' - (U^2)'' = 0. \quad (2.15)$$

Balancing the highest linear term with nonlinear terms in equation (2.15), we have the expansion

$$U = a_0 + a_1\varphi + a_2\varphi^2, \quad V = b_0 + b_1\varphi + b_2\varphi^2. \quad (2.16)$$

Substituting (2.16) into (2.15) yields

$$-\alpha^2a_0 + \beta a_0 + 2k^2a_2 + a_0b_0 = 0,$$
$$2ka_1 - \alpha^2a_1 + \beta a_1 + a_1b_0 + a_0b_1 = 0,$$
$$8ka_2 - \alpha^2a_2 + \beta a_2 + a_2b_0 + a_1b_1 + a_0b_2 = 0,$$
$$2a_1 + a_2b_1 + a_1b_2 = 0,$$
$$6a_2 + a_2b_2 = 0,$$
$$-2ka_0a_1 + 2k^2b_1 + 2\alpha kb_1 + 6kb_0b_1 = 0,$$
$$-2ka_1^2 - 4ka_0a_2 + 6kb_1^2 + 16k^2b_2 + 4\alpha kb_2 + 12kb_0b_2 = 0,$$
$$-2a_0a_1 - 6ka_1a_2 + 8kb_1 + 2\alpha b_1 + 6b_0b_1 + 18kb_1b_2 = 0,$$
$$-2a_1^2 - 4a_0a_2 - 4ka_2^2 + 6b_1^2 + 40kb_2 + 4\alpha b_2 + 12b_0b_2 + 12kb_2^2 = 0,$$
$$-6a_1a_2 + 6b_1 + 18b_1b_2 = 0,$$
$$-4a_2^2 + 24b_2 + 12b_2^2 = 0.$$

From the output of *Mathematica*, we obtain two solutions, namely

$$a_0 = \pm 2\sqrt{2}k, \quad a_1 = b_1 = 0, \quad a_2 = \pm 6\sqrt{2},$$
$$b_0 = -\frac{1}{3}(8k + \alpha), b_2 = -6, \beta = \frac{1}{3}(3\alpha^2 + \alpha - 10k) \quad (2.17)$$

with k, α being arbitrary constants and

$$a_0 = b_1 = a_2 = 0, \quad b_0 = -\frac{1}{12}(16k + 4\alpha + a_1^2),$$
$$b_2 = -2, \quad \beta = \frac{1}{12}(12\alpha^2 + 4\alpha - 8k + a_1^2), \quad (2.18)$$

with k, a_1 and α being arbitrary constants.

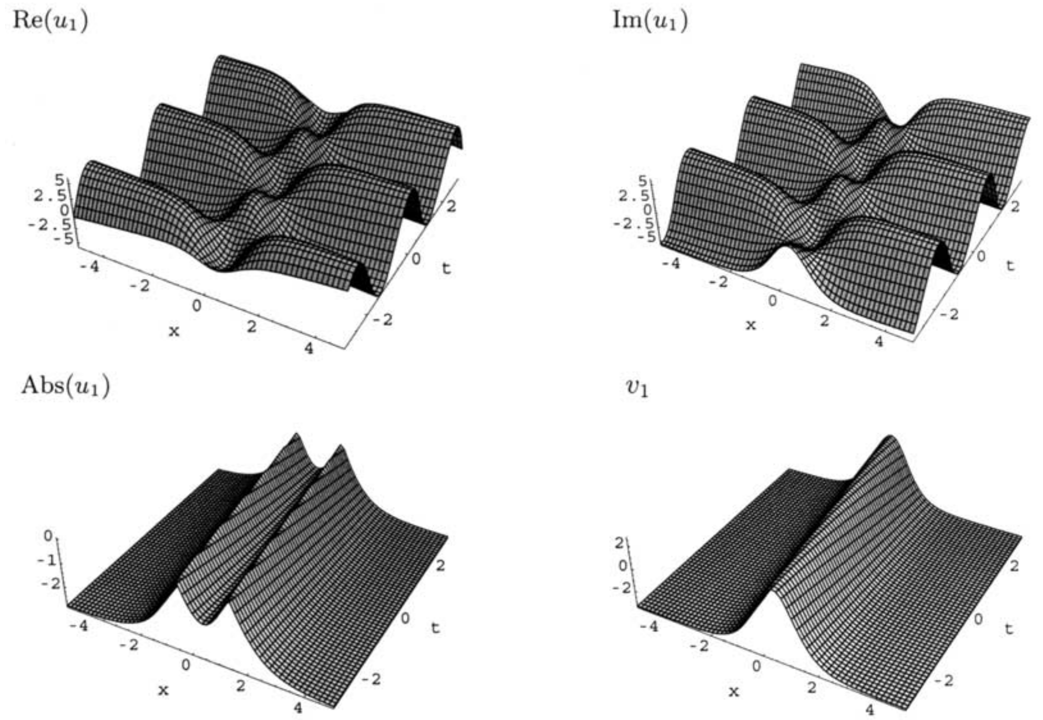

Fig. 2. The soliton solutions u_1 and v_1 with two-peak shaped for u_1 and bell-shaped for v_1, where $k = 0.5, p = 0.1$.

From (1.5)-(1.7) and (2.17), we get three kinds of solutions:

$$u_1 = \pm 2\sqrt{2}ke^{i\theta}[1 - 3\tanh^2\sqrt{-k}\xi],$$
$$v_1 = -\frac{1}{3}(8k + \alpha) + 6k\tanh^2\sqrt{-k}\xi, \quad k < 0,$$

$$u_2 = \pm 2\sqrt{2}ke^{i\theta}[1 + 3\tan^2\sqrt{k}\xi],$$
$$v_2 = -\frac{1}{3}(8k + \alpha) - 6k\tan^2\sqrt{k}\xi, \quad k > 0,$$

where $\xi = x + 2\alpha t, \ \ \theta = \alpha x + \frac{1}{3}(3\alpha^2 + \alpha - 10k)t$, and

$$u_3 = \pm\frac{6\sqrt{2}}{\xi^2}e^{i\theta}, \ \ v_3 = -\frac{1}{3}\alpha - \frac{6}{\xi^2}, \ \ k = 0,$$

where $\xi = x + 2\alpha t, \ \ \theta = \alpha x + \frac{1}{3}(3\alpha^2 + \alpha)t$.

Again from (1.5)-(1.7) and (2.18), we obtain other three kinds of solutions

$$u_4 = -a_1\sqrt{-k}e^{i\theta}\tanh\sqrt{-k}\xi,$$
$$v_4 = -\frac{1}{12}(16k + 4\alpha + a_1^2) + 2k\tanh^2\sqrt{-k}\xi, \ \ k < 0,$$

$$u_5 = a_1\sqrt{k}e^{i\theta}\tan\sqrt{k}\xi,$$
$$v_5 = -\frac{1}{12}(16k + 4\alpha + a_1^2) - 2k\tan^2\sqrt{k}\xi, \ \ k > 0,$$

where $\xi = x + 2\alpha t, \ \theta = \alpha x + \frac{1}{12}(12\alpha^2 + 4\alpha - 8k + a_1^2)t$, and

$$u_6 = -\frac{a_1}{\xi}e^{i\theta}, \ \ v_6 = -\frac{1}{12}(4\alpha + a_1^2) - \frac{2}{\xi^2}, \ \ k = 0,$$

where $\xi = x + 2\alpha t, \ \theta = \alpha x + \frac{1}{12}(12\alpha^2 + 4\alpha + a_1^2)t$. The properties of the solutions u_i and v_i $(i = 1, 2, \cdots, 6)$ are shown in Figures 2 - 7.

Example 5. Consider the (2+1)-dimensional Davey-Stewartson equation [17]

$$\begin{aligned} &iu_t + u_{xx} - u_{yy} - 2|u|^2u - 2uv = 0, \\ &v_{xx} + v_{yy} + 2(|u|^2)_{xx} = 0. \end{aligned} \tag{2.19}$$

Similar to the above Example 4, we introduce the transformations

$$\begin{aligned} &u = e^{i\theta}U(\xi), v = V(\xi), \\ &\theta = \alpha x + \beta y + \delta t, \ \ \xi = x + \gamma y + \lambda t, \end{aligned} \tag{2.20}$$

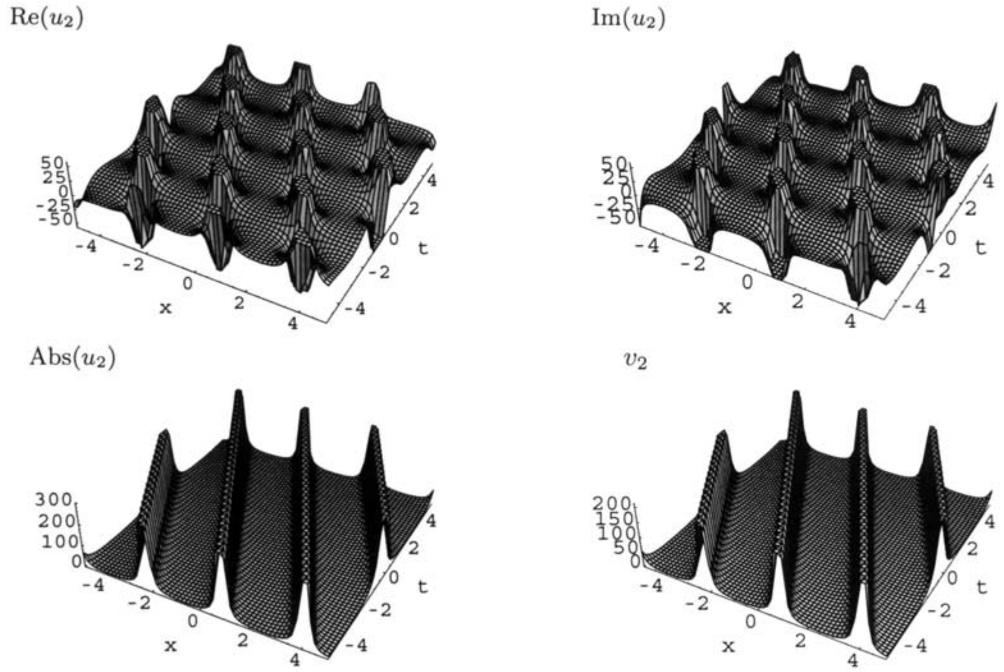

Fig. 3. The periodic solutions u_2 and v_2, where $k = 1, p = 0.25$.

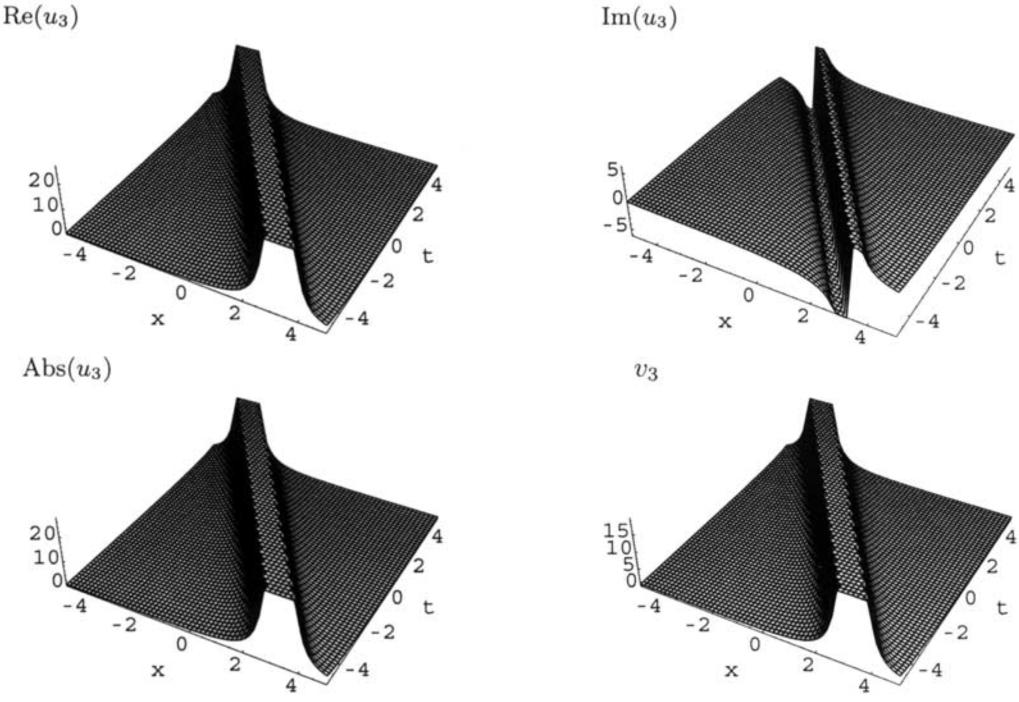

Fig. 4. The rational solutions u_3 and v_3, where $k = 0.5, p = 0.3$.

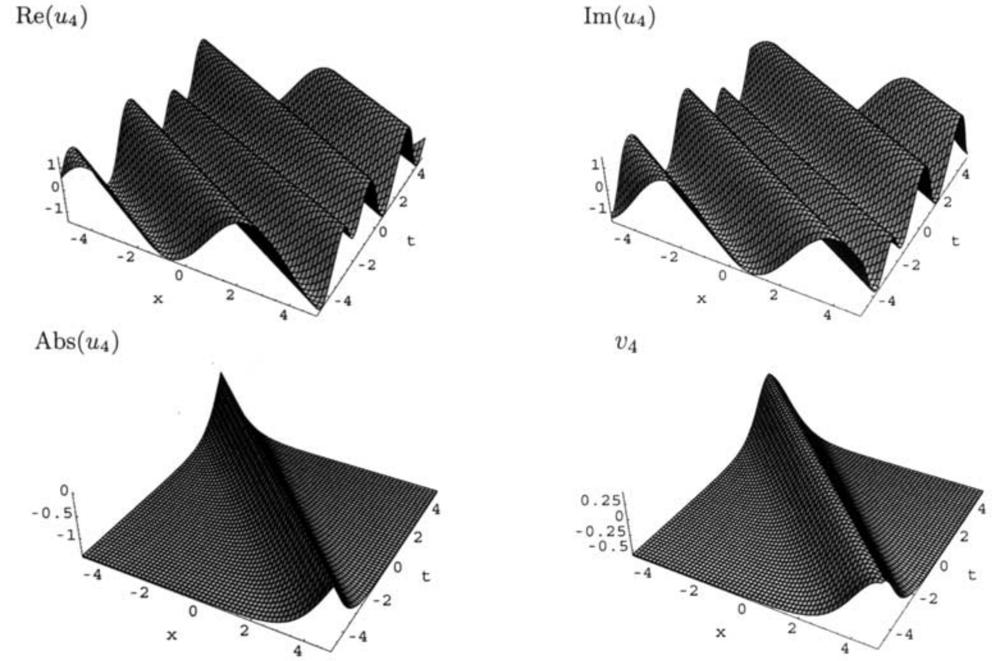

Fig. 5. The soliton solutions u_4 and v_4 with all bell-shaped for u_4 and v_4, where $k = -0.5, p = 0.5, a_1 = 1$.

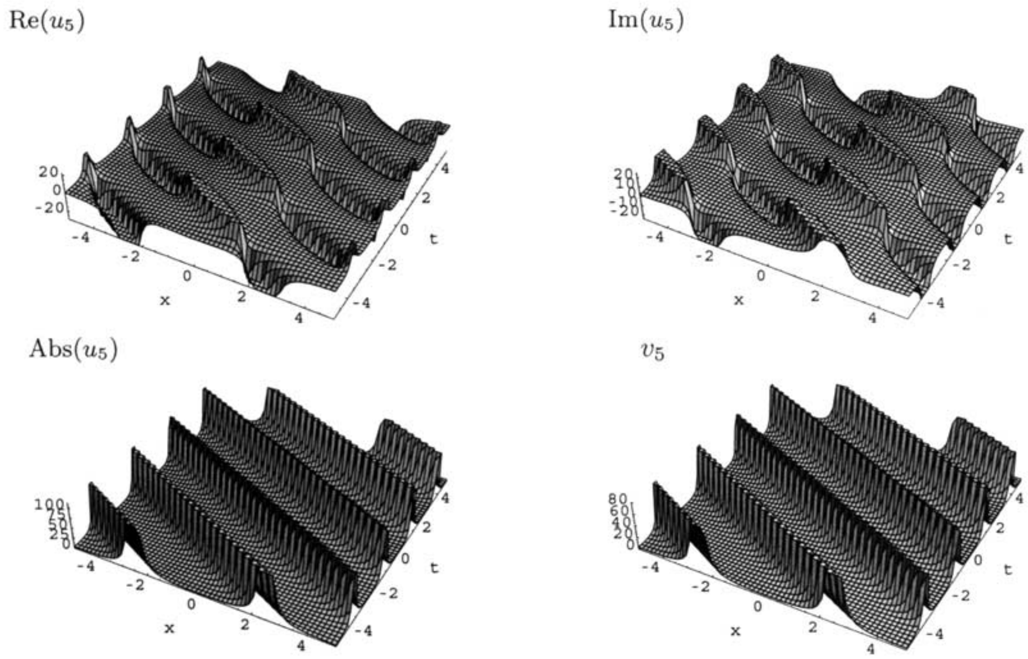

Fig. 6. The periodic solutions u_5 and v_5, where $k = 0.25, p = 0.2, a_1 = 1$.

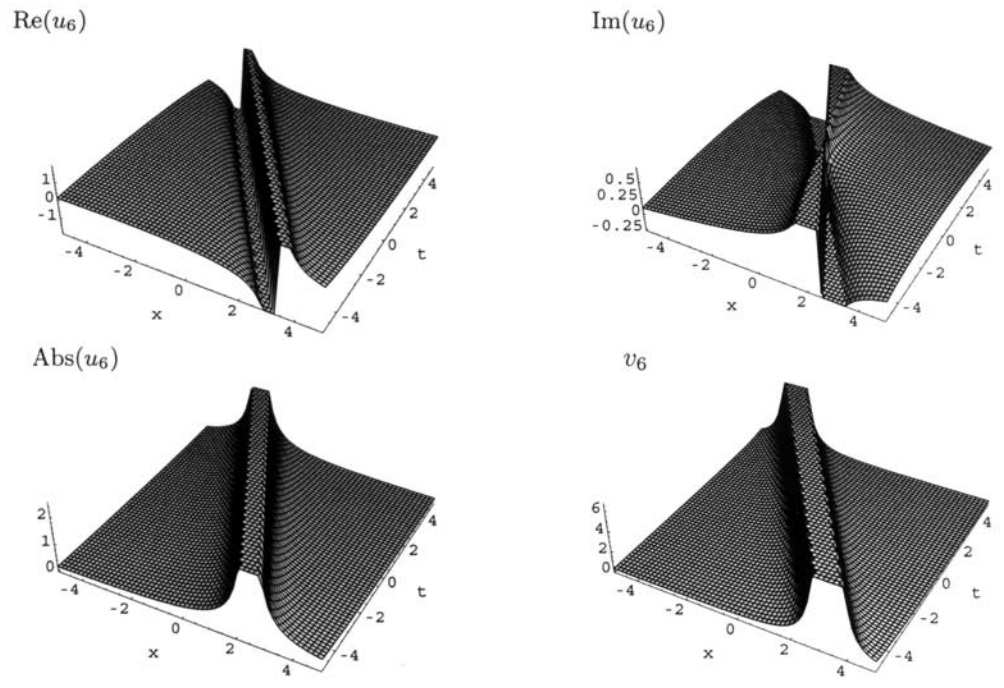

Fig. 7. The rational solutions u_6 and v_6, where $p = 0.3$, $a_1 = 1$.

where α, β, δ, γ, λ are real constants and $U(\xi)$, $V(\xi)$ are real functions.

Substituting (2.20) into (2.19), we find that $\lambda = -2(\alpha - \beta\gamma)$. U and V satisfy the system

$$(\beta^2 - \alpha^2 - \delta)U + (1 - \gamma^2)U'' - 2U^3 - 2UV = 0,$$
$$(1 + \gamma^2)V'' + (V^2)'' = 0. \quad (2.21)$$

Balancing the highest linear term and nonlinear terms in (2.21) we have the ansatz

$$U = a_0 + a_1\varphi, \quad V = b_0 + b_1\varphi + b_2\varphi^2. \quad (2.22)$$

Substituting (2.22) into (2.21), we obtain the following system of algebraic equations:

$$(\beta^2 - \alpha^2 - \delta)a_0 - 2a_0^3 - 2a_0b_0 = 0,$$
$$2ka_1 - 2\gamma^2 ka_1 + (\beta^2 - \alpha^2 - \delta)a_1 - 6a_0^2a_1 - 2a_1b_0 - 2a_0b_1 = 0,$$
$$-6a_0a_1^2 - 2a_1b_1 - 2a_0b_2 = 0,$$
$$2a_1 - 2\gamma^2 a_1 - 2a_1^3 - 2a_1b_2 = 0,$$
$$4k^2a_1^2 + 2k^2b_2 + 2k^2\gamma^2 b_2 = 0,$$
$$8ka_0a_1 + 2kb_1 + 2k\gamma^2 b_1 = 0,$$
$$16ka_1^2 + 8kb_2 + 8k\gamma^2 b_2 = 0,$$
$$8a_0a_1 + 2b_1 + 2\gamma^2 b_1 = 0,$$
$$12a_1^2 + 6b_2 + 6\gamma^2 b_2 = 0.$$

Solving the system by *Mathematica*, we find two kinds of solutions:

$$b_0 = \frac{1}{2}(\beta^2 - \alpha^2 - \delta - 2a_0^2), \quad b_1 = -2a_0a_1,$$
$$b_2 = -a_1^2, \quad \gamma = \pm 1 \quad (2.23)$$

with k, α, β, δ, a_0 and a_1 being arbitrary constants and

$$a_0 = b_1 = 0, \quad b_0 = \frac{1}{2}(\beta^2 - \alpha^2 - \delta + 2k - 2\gamma^2 k),$$
$$a_1 = \pm i\sqrt{1 + \gamma^2}, \quad b_2 = 2 \quad (2.24)$$

with k, α, β, γ and δ being arbitrary constants.

From (1.5)-(1.7) and (2.23) we obtain three kinds of solutions

$$u_1 = e^{i\theta}[a_0 - a_1\sqrt{-k}\tanh\sqrt{-k}\xi],$$
$$v_1 = \frac{1}{12}(\beta^2 - \alpha^2 - \delta - 2a_0^2) + 2a_0a_1\tanh\sqrt{-k}\xi + a_1^2 k\tanh^2\sqrt{-k}\xi, \quad k < 0,$$

$$u_2 = e^{i\theta}[a_0 + a_1\sqrt{k}\tan\sqrt{-k}\xi],$$
$$v_2 = \frac{1}{12}(\beta^2 - \alpha^2 - \delta - 2a_0^2) - 2a_0a_1\tan\sqrt{k}\xi - a_1^2k\tan^2\sqrt{k}\xi, \ \ k > 0,$$

$$u_3 = e^{i\theta}\left[a_0 - \frac{a_1}{\xi}\right],$$
$$v_3 = \frac{1}{12}(\beta^2 - \alpha^2 - \delta - 2a_0^2) + \frac{2a_0a_1}{\xi} - \frac{a_1^2}{\xi^2}, \ \ k = 0,$$

where $\xi = x + \pm y - 2(\beta \pm \alpha)t, \ \ \theta = \alpha x + \beta y + \delta t.$

Again from (1.5)-(1.7) and (2.24), we also find three kinds of solutions

$$u_4 = \pm ie^{i\theta}\sqrt{-k(1+\gamma^2)}\tanh\sqrt{-k}\xi,$$
$$v_4 = \frac{1}{12}(\beta^2 - \alpha^2 - \delta + 2k - 2\gamma^2k) - 2k\tanh^2\sqrt{-k}\xi, \ \ k < 0,$$

$$u_5 = \pm ie^{i\theta}\sqrt{k(1+\gamma^2)}\tan\sqrt{k}\xi,$$
$$v_5 = \frac{1}{12}(\beta^2 - \alpha^2 - \delta + 2k - 2\gamma^2k) + 2k\tan^2\sqrt{k}\xi, \ \ k > 0,$$

$$u_6 = \pm\frac{i\sqrt{1+\gamma^2}}{\xi}e^{i\theta},$$
$$v_6 = \frac{1}{12}(\beta^2 - \alpha^2 - \delta) + \frac{2}{\xi^2}, \ \ k = 0,$$

where $\xi = x + \gamma y - 2(\alpha - \beta\gamma)t, \ \ \theta = \alpha x + \beta y + \delta t.$

In summary, we have applied the generalized tanh method to construct a series of travelling wave solutions for some special types of equations. In the fact, the present method is readily applicable to a large variety of such nonlinear equations. The obtained solutions include soliton solutions, periodic solutions and rational solutions. The properties of the solutions are shown in Figures 1 - 7. The physical relevance of soliton solutions and periodic solutions is clear to us. The rational solutions are a disjoint union of manifolds, and the particle system of KdV and Boussinesq was analyzed in [19 - 21]. We also can see that some solutions obtained in this paper develop a singularity at a finite point, i. e. for any fixed $t = t_0$, there is an x_0 at which these solutions blow-up. There is current interest in the formulation of so-called "hot-spots" or "blow-ups" of solutions [22 - 25]. It appears that these singular solutions will model such physical phenomena.

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[1] W. Malfliet, Amer. J. Phys. **60**, 650 (1992).
[2] E. J. Parkes, J. Phys. A **27**, L497 (1994).
[3] E. J. Parkes and B. R. Duffy, Computer Phys. Commun. **98**, 288 (1996).
[4] B. R. Duffy and E. J. Parkes, Phys. Lett. A **214**, 271 (1996).
[5] E. J. Parkes and B. R. Duffy, Phys. Lett. A **229**, 217 (1997).
[6] Y. T. Gao, and B. Tian, Computers Math. Applic. **33**, 115 (1997).
[7] B. Tian and Y. T. Gao, Comput. Phys. Commun, **95**, 139 (1996).
[8] Z. B. Li and S. Q. Zhang, Acta. Math. Sin. **17**, 81 (1997).
[9] E. G. Fan, and Q. H. Zhang, Phys. Lett. A **246**, 403 (1998).
[10] E. G. Fan, Phys. Lett. A **277**, 212 (2000).
[11] E. G. Fan, Z. Naturforsch. **56a**, 312 (2001).
[12] E. G. Fan, Phys. Lett. A **282**, 18 (2001).
[13] E. G. Fan, J. Zhang, and Y. C. Hon, Phys. Lett. A **291**, 376 (2001).
[14] Z. Jiang, Inverse Probl. **4**, 349 (1989).
[15] M. Lakshmanan and P. Kaliappan, J. Math. Phys. **24**, 795 (1983).
[16] R. K. Dodd, and R. K. Bullough, Proc. Roy. Soc. A **3511** 499 (1976).
[17] C.H. Gu, H. S. Hu, and Z. X. Zhou, Darboux Transformations in Solition Theory and its Geometric Applications, Shanghai Sci. Tech. Publ. 1999.
[18] T. Yoshinaga, M. Wakamiya, and T. Kakutani, Phys. Fluids, **A 3**, 83 (1991).
[19] M. Airault, H. Mckean, and J. Moser, Commun. Pure. Appl. Math. **30**, 95 (1977).
[20] M. Adler and J. Moser, Commun. Math. Phys. **19**, 1 (1978).
[21] A. Nakamura and R. Hirota, J. Phys. Soc. Jpn. **54**, 491 (1985).
[22] R. L. Sachs, Physica D **30**, 1 (1988).
[23] C. J. Coleman, J. Aust. Math. Soc. Ser. B, **33**, 1 (1992).
[24] N. F. Smyth, J. Aust. Math. Soc. Ser. B, **33**, 403 (1992).
[25] P. A. Clarkson, and E. L. Mansfield, Physica D **70**, 250 (1993).